Sec. 3.4 Solving Systems of Linear Equations in Three Variables

A system of *linear* equations is any system whose equations only contain constant or linear terms. Each term can only have one variable (or no variable), and its power can only be 1.

A system of equations *in three variables* is any system that essentially contains three unknown quantities. The variables x, y, and z are usually used to represent these unknown values.

Needless to say, a *system of linear equations in three variables* is a system that meets both conditions listed above. While a system of equations can contain any number of equations, ones with three unknown quantities usually require three equations (special cases might require only two, and additional conditions might require more than three). For the purposes of this lesson, all systems can be written in the general form.

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

where a_n, b_n, c_n, and d_n are any real number.

To solve such a system means to find the coordinate triple (x, y, z) that makes all three equations a true statement.
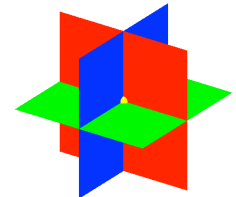
A Graphical Representation

Whereas linear equations in two variables are graphed as lines in the two-dimensional Cartesian coordinate plane, linear equations in three variables require three-dimensional space to be graphed. These 3D equations create planes.

While it is not the intent of this lesson to graph planes in space, it is important to understand how the graphs of the three-variable equations planes behave in space. Just like the systems of two linear equations, there are three scenarios.
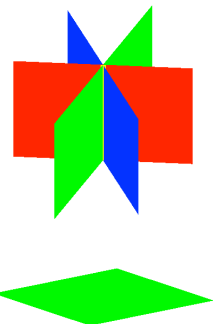
One Solution

In most cases, the graphs of the three equations (three planes) will intersect at a single (x, y, z) coordinate in space. This would mean that there is one solution to the consistent, independent system.

A Graphical Representation (continued)

Infinitely Many Solutions

It is possible for the planes to intersect not at a point, but rather along a line. When this occurs, there are infinitely many solutions since any (x, y, z) coordinate on the line would be at the intersection of the three planes. The system would be described as consistent, but dependent.

In rare instances, the three equations could graph to be the same plane. This is analogous to the 2D coinciding lines and would also result in infinitely many solutions since any (x, y, z) coordinate in one equation would work for any other equation. The system would be described as consistent and dependent.

No Solution

If the three planes are parallel to each other, they would never intersect. This, of course, means there would be no solution to the system. It is possible, however, for each plane to intersect the other two, but there not be a common (x, y, z) coordinate for all three. In both of these situations, the system would be described as inconsistent.
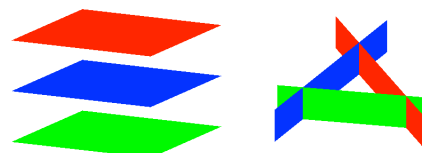
Solving Algebraically

As with systems of equations in two variables, there are many methods for solving systems in three variables. If technology is present, using matrices is generally the quickest and most efficient. Otherwise, a combination of the elimination and substitution methods works well. The steps outlined here illustrate how one might solve a general system of three linear equations in three variables. Keep in mind, however, that choosing certain equations in each step might provide shortcuts.

Here is the basic plan:
- Select a pair of equations and eliminate one variable.
- Select a different pair of equations and eliminate the same variable.
- Use the resulting equations (with the eliminated variable) to write a system of equations in two variables.
- Solve this two-variable system using any method to find the two variables.
- Substitute the two values into one of the original equations to find the remaining variable's value.

Note that if you ever get a false statement, there will be no solution. If you get a true statement (an identity), there will be infinitely many solutions.

Example 1 – One Solution

Solve the system $\begin{cases} 3x + 4y + 2z = 11 \\ 2x + 3y - z = 4 \\ 5x + 5y - 3z = -1 \end{cases}$.

Select the first and second equations to eliminate the z variable (multiply the second equation by 2 and add).

$$\begin{cases} 3x + 4y + 2z = 11 \\ 4x + 6y - 2z = 8 \end{cases}$$
$$7x + 10y = 19$$

Select the second and third equations to eliminate the same z variable (multiply the second equation by -3 and add).

$$\begin{cases} -6x - 9y + 3z = -12 \\ 5x + 5y - 3z = -1 \end{cases}$$
$$-x - 4y = -13$$

You now have two equations in terms of x and y.

$$\begin{cases} 7x + 10y = 19 \\ -x - 4y = -13 \end{cases}$$

Use any method to solve this system for x and y (elimination is shown here).

$$\begin{cases} 14x + 20y = 38 \\ -5x - 20y = -65 \end{cases} \qquad \begin{cases} 7x + 10y = 19 \\ -7x - 28y = -91 \end{cases}$$
$$\begin{array}{cc} 9x = -27 & -18y = -72 \\ x = -3 & y = 4 \end{array}$$

Substitute $x = 3$ and $y = 4$ into the second original equation and solve for z.

$$2(-3) + 3(4) - z = 4$$
$$z = 2$$

Therefore, the solution to the consistent, independent system is (−3, 4, 2).

Note: You can verify this solution by plugging it into the first and third original equations.

Example 2 – No Solution

Solve the system $\begin{cases} x - 2y + z = 4 \\ x + y + z = 2 \\ 3x + 3y + 3z = 14 \end{cases}$.

Select the first and second equations to eliminate the z variable (multiply the second equation by -1 and add).

$$\begin{cases} x - 2y + z = 4 \\ \underline{-x - y - z = -2} \end{cases}$$
$$-3y = 2$$
$$y = -2/3$$

Notice in this case, the result was an equation in terms of just one variable (instead of two). Although this solves for y, you still need to repeat the process to find x. Select the first and third equations to eliminate the same z variable (multiply the first equation by -3 and add).

$$\begin{cases} -3x + 6y - 3z = -12 \\ \underline{3x + 3y + 3z = 14} \end{cases}$$
$$9y = 2$$
$$y = 2/9$$

Again, you are left with an equation in terms of just y. However, this time $y = 2/9$ instead of $-2/3$. This means there cannot be one solution, but it is not obvious if there are no solutions or an infinite number of solutions. When this occurs, it is usually easiest to start over using different equations and/or variables. Let's try multiplying the second original equation by -3 and adding to the third equation.

$$\begin{cases} -3x - 3y - 3z = -6 \\ \underline{3x + 3y + 3z = 14} \end{cases}$$
$$0 = 8$$

Since this is clearly a false statement, we now know that this inconsistent system of equations has no solution.

Note: Instead of starting over after the contradictory y values, you could have substituted one of them into two equations and tried to solve for x and z. You would have eventually ended up with a false statement as well. Doing so in this problem, however, would have caused several fractions. Starting over was easier in terms of computation.

Example 3 – Infinitely Many Solutions

$$\text{Solve the system } \begin{cases} x + y + z = 2 \\ x + y - z = 2 \\ 2x + 2y + z = 4 \end{cases}.$$

Select the first and second equations to eliminate the z variable (just add).

$$\begin{cases} x + y + z = 2 \\ x + y - z = 2 \end{cases}$$
$$\overline{\quad 2x + 2y = 4 \quad}$$

Select the second and third equations to eliminate the same z variable (again, just add).

$$\begin{cases} x + y - z = 2 \\ 2x + 2y + z = 4 \end{cases}$$
$$\overline{\quad 3x + 3y = 6 \quad}$$

You now have two equations in terms of x and y.

$$\begin{cases} 2x + 2y = 4 \\ 3x + 3y = 6 \end{cases}$$

Use elimination with this smaller system (multiply the top equation by 3 and the bottom equation by –2).

$$\begin{cases} 6x + 6y = 12 \\ -6x - 2y = -12 \end{cases}$$
$$\overline{\quad 0 = 0 \quad}$$

The result is an identity, so there must be an infinite number of solutions to the smaller system. As a result, there are an infinite number of solutions to the original consistent, dependent system.

Dependent Solutions

When a system of equations has an infinite number of solutions, it is described as dependent. This is because one or two of the unknown quantities can be found only if the other value(s) are known. In other words, x and y might be dependent on the value of z.

It is often helpful to describe the dependent relationship among the variables. When possible, it is common to describe x and y in terms of z.

To describe the relationship, do the following:
- Choose two equations and eliminate y. Solve for x in terms of z.
- Choose two equations and eliminate x. Solve for y in terms of z.
- State the solution as $(x(z), y(z),$ and $z)$.

Example 4 – Describing Solutions in a Dependent System

Describe the solutions of the dependent the system $\begin{cases} x+y+z=2 \\ x+y-z=2 \\ 2x+2y+z=4 \end{cases}$.

If you multiply the second equation by -1 and add it to the first equation, you get

$$\begin{cases} x+y+z=2 \\ \underline{-x-y+z=-2} \end{cases}$$
$$2z=0$$
$$z=0$$

Since $z = 0$, z cannot be written in terms of x or y in this example. Therefore, choose another variable to be the one the other two are dependent upon. Let's choose x. Add the first and second equations down.

$$\begin{cases} x+y+z=2 \\ \underline{x+y-z=2} \end{cases}$$
$$2x+2y=4$$

Now solve this equation for y in terms of x.

$$y=2-x$$

This shows that y is dependent on x. Since $z = 0$ and $y = 2 - x$, we can write the dependent relationship as $(x, 2 - x, 0)$.

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Chapter 3
Systems in Three Variables Worksheet

Name   _____
Date   _____
Period   _____

Solve the following systems of equations. If the system is dependent, describe the relationship among the variables.

1. $\begin{cases} 4x - 2y + z = 8 \\ \quad\;\; -y + z = 4 \\ \quad\quad\quad\; 2z = 4 \end{cases}$

2. $\begin{cases} 2x + 4y + z = 1 \\ x - 2y - 3z = 2 \\ x + y - z = -1 \end{cases}$

3. $\begin{cases} 2x + y - z = 7 \\ x - 2y + 2z = -9 \\ 3x - y + z = 5 \end{cases}$

4. $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ 3x - z = 0 \end{cases}$

5. $\begin{cases} 3x - 3y + 6z = 6 \\ 5x - 8y + 13z = 7 \\ x + 2y - z = 5 \end{cases}$